

Analytical solutions of the D -dimensional Schrödinger equation with the Woods-Saxon potential for arbitrary l state

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Abstract

In this work, the analytical solution of the hyper-radial Schrödinger equation for the spherical Woods-Saxon potential in D dimensions is presented. In our calculations, we have applied the Nikiforov-Uvarov method by using the Pekeris approximation to the centrifugal potential for arbitrary l states. The bound state energy eigenvalues and corresponding eigenfunctions are obtained for various values of n and l quantum numbers.

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I. INTRODUCTION

An analytical solution of the radial Schrödinger equation is of high importance in nonrelativistic quantum mechanics, because the wave function contains all necessary information for full description of a quantum system. There are only few potentials for which the radial Schrödinger equation can be solved explicitly for all n and l quantum numbers. So far, many methods were developed, such as supersymmetry (SUSY) [1,2] and the Pekeris approximation [3-8], to solve the radial Schrödinger equation exactly or quasi-exactly for $l \neq 0$ within these potentials. Levai and Williams suggested a simple method for constructing potentials for which the Schrödinger equation can be solved exactly in terms of special functions [9] and showed relationship between the introduced formalism and supersymmetric quantum mechanics [1]. Amore *et al.* presented a new method [10] for the solution of the Schrödinger equation applicable to problems of nonperturbative nature. In addition, they applied the method to the quantum anharmonic oscillator and found energy eigenvalues and wave functions, even for strong couplings.

The three-dimensional radial Schrödinger equation for the spherical Woods-Saxon potential [11] can not be solved analytically for $l \neq 0$ states, because of the centrifugal term $\sim r^{-2}$. However, Flügge gave an exact expression for the wave function, but a graphical method was suggested for the energy eigenvalues at $l = 0$ [4]. It is well known that the Woods-Saxon potential is one of the important short-range potentials in physics. Furthermore, this potential were applied to numerous problems, in nuclear and particle physics, atomic physics, condensed matter, and chemical physics.

Recently, an alternative method known as the Nikiforov-Uvarov (NU) method [12] was proposed for solving the Schrödinger equation. Therefore, it would be interesting and important to solve the nonrelativistic radial Schrödinger equation for Woods-Saxon potential for $l \neq 0$, since it has been extensively used to describe the bound and continuum states of the interacting systems. Thus, one needs to obtain the energy eigenvalues and corresponding eigenfunctions of the one particle problem within this potential. The NU method was used by C. Berkdemir et al. [13] to solve the radial Schrödinger equation for the generalized Woods-Saxon potential for $l = 0$. However, in this work, the authors made errors in application of the NU method, which led to no-correct results [14]. In Refs.[7,8] the radial Schrödinger equation with the spherical Woods-Saxon potential have been solved using

the Nikiforov-Uvarov method and Pekeris approximation for arbitrary l states. The same method has been employed to solve the Klein-Gordon equation with the spherical Woods-Saxon potential for $l \neq 0$ states Ref.[15].

In this work, we solve the D dimensional hyper-radial Schrödinger equations with spherical Woods-Saxon potential for arbitrary l states. Here the NU method [12] and the Pekeris approximation is applied to find energy eigenvalues and corresponding eigenfunctions of the considering problem.

II. NIKIFOROV-UVAROV METHOD

The Nikiforov-Uvarov (NU) method is based on the solutions of general second-order linear equations with special orthogonal functions. It has been extensively used to solve the nonrelativistic Schrödinger equation and other Schrödinger-like equations. The one-dimensional Schrödinger equation or similar second-order differential equations can be written using NU method in the following form:

$$\psi''(z) + \frac{\tilde{\tau}(z)}{\sigma(z)}\psi'(z) + \frac{\tilde{\sigma}(z)}{\sigma^2(z)}\psi(z) = 0, \quad (2.1)$$

where $\sigma(z)$ and $\tilde{\sigma}(z)$ are polynomials, at most second-degree, and $\tilde{\tau}(z)$ is a first-degree polynomial.

Using Eq.(2.1) the transformation

$$\psi(z) = \Phi(z)y(z) \quad (2.2)$$

one reduces it to the hypergeometric-type equation

$$\sigma(z)y'' + \tau(z)y' + \lambda y = 0. \quad (2.3)$$

The function $\Phi(z)$ is defined as the logarithmic derivative [12]

$$\frac{\Phi'(z)}{\Phi(z)} = \frac{\pi(z)}{\sigma(z)}, \quad (2.4)$$

where $\pi(z)$ is at most the first-degree polynomial.

The another part of $\psi(z)$, namely $y(z)$, is the hypergeometric-type function, that for fixed n is given by the Rodriguez relation:

$$y_n(z) = \frac{B_n}{\rho(z)} \frac{d^n}{dz^n} [\sigma^n(z)\rho(z)], \quad (2.5)$$

where B_n is the normalization constant and the weight function $\rho(z)$ must satisfy the condition [12]

$$\frac{d}{dz}(\sigma(z)\rho(z)) = \tau(z)\rho(z), \quad (2.6)$$

with $\tau(z) = \tilde{\tau}(z) + 2\pi(z)$.

For accomplishment of the conditions imposed on function $\rho(z)$, the classical orthogonal polynomials, it is necessary, that polynomial $\tau(z)$ becomes equal to zero in some point of an interval (a, b) and derivative of this polynomial for this interval at $\sigma(z) > 0$ will be negative, i.e. $\tau'(z) < 0$.

The function $\pi(z)$ and the parameter λ required for this method are defined as follows:

$$\pi(z) = \frac{\sigma' - \tilde{\tau}}{2} \pm \sqrt{\left(\frac{\sigma' - \tilde{\tau}}{2}\right)^2 - \tilde{\sigma} + k\sigma}, \quad (2.7)$$

$$\lambda = k + \pi'(z). \quad (2.8)$$

On the other hand, in order to find the value of k , the expression under the square root must be the square of a polynomial. This is possible only if its discriminant is zero. Thus, the new eigenvalue equation for the Schrödinger equation becomes [12]

$$\lambda = \lambda_n = -n\tau' - \frac{n(n-1)}{2}\sigma'', (n = 0, 1, 2, \dots). \quad (2.9)$$

After the comparison of Eq.(2.8) with Eq.(2.9), we obtain the energy eigenvalues.

III. SOLUTIONS OF THE SCHRÖDINGER EQUATION WITH THE WOODS-SAXON POTENTIAL

Woods and Saxon introduced a potential to study elastic scattering of 20 MeV protons by a heavy nuclei [11]. The spherical Woods-Saxon potential that was used as a major part of nuclear shell model, has received a lot of attention in nuclear mean field model. The spherical standard Woods-Saxon potential [11] is defined by

$$V(r) = -\frac{V_0}{1 + \exp\left(\frac{r-R_0}{a}\right)}, a \ll R_0. \quad (3.1)$$

This potential was used for description of interaction of a neutron with a heavy nucleus. The parameter R_0 is interpreted as radius of a nucleus, the parameter a characterizes thickness of the superficial layer inside, which the potential falls from value $V = 0$ outside of a nucleus

up to value $V = -V_0$ inside a nucleus. At $a = 0$, one gets the simple potential well with jump of potential on the surface of a nucleus.

Using D -dimensional ($D \geq 2$) polar coordinates with polar variable r (hyperradius) and angular variables $\theta_1, \theta_2, \dots, \theta_{D-2}, \phi$ (hyperangles), the Laplasian operator in polar coordinates $(r, \theta_1, \theta_2, \dots, \theta_{D-2}, \phi)$ of R^D is

$$\nabla_D^2 = r^{1-D} \frac{\partial}{\partial r} \left(r^{D-1} \frac{\partial}{\partial r} \right) + \frac{\Lambda_D^2}{r^2},$$

where Λ_D^2 is a partial differential operator on the unit sphere S^{D-1} (Laplace-Beltrami operator, or grand orbital operator, or hyperangular momentum operator) defined analogously to a three-dimensional angular momentum, Avery [16].

The D dimensional Schrödinger equation with spherically symmetric potential $V(r)$ has the form [16]

$$\left(-\frac{\hbar^2}{2\mu} \nabla_D^2 + V(r) - E_{nl} \right) \psi_{nlm}(r, \Omega_D) = 0, \quad (3.2)$$

where μ is the reduced mass, \hbar is the Planck's constant and

$$\psi_{nlm}(r, \Omega_D) = R_{nl}(r) Y_{lm}(\Omega_D). \quad (3.3)$$

The Laplasian operator divides into a hyper-radial part $r^{1-D} \frac{\partial}{\partial r} \left(r^{D-1} \frac{\partial}{\partial r} \right)$ and an angular part $\frac{\Lambda_D^2}{r^2} = -\frac{\hat{L}_D^2}{\hbar^2 r^2}$, i.e.

$$\nabla_D^2 = r^{1-D} \frac{\partial}{\partial r} \left(r^{D-1} \frac{\partial}{\partial r} \right) - \frac{\hat{L}_D^2}{\hbar^2 r^2}, \quad (3.4)$$

where \hat{L}_D is the grand orbital angular momentum operator. The eigenfunctions of \hat{L}_D^2 are the hyper-spherical harmonics $Y_{lm}(\Omega_D)$

$$L_D^2 Y_{lm}(\Omega_D) = \hbar^2 l(l + D - 2) Y_{lm}(\Omega_D), \quad (3.5)$$

where l is the angular momentum quantum number.

After substituting the Eqs.(3.3), (3.4), (3.5) into (3.2) and using the fact that $\psi_{nlm}(r, \Omega_D)$ is the eigenfunction of \hat{L}_D^2 with eigenvalue $\hbar^2 l(l + D - 2)$, we obtain an equation known as the hyper-radial Schrödinger equation with Woods-Saxon potential

$$\frac{d^2 R_{nl}(r)}{dr^2} + \frac{D-1}{r} \frac{dR_{nl}(r)}{dr} + \frac{2\mu}{\hbar^2} \left[E + \frac{V_0}{1 + \exp\left(\frac{r-R_0}{a}\right)} - \frac{\hbar^2 l(l + D - 2)}{2\mu r^2} \right] R_{nl}(r) = 0, \quad (0 \leq r < \infty). \quad (3.6)$$

Introducing a new function

$$u_{nl}(r) = r^{\frac{D-1}{2}} R_{nl}(r),$$

Eq.(3.6) reduces to

$$\frac{d^2 u_{nl}(r)}{dr^2} + \frac{2\mu}{\hbar^2} \left[E + \frac{V_0}{1 + \exp\left(\frac{r-R_0}{a}\right)} - \frac{\hbar^2 \left(l + \frac{D-1}{2}\right) \left(l + \frac{D-3}{2}\right)}{2\mu r^2} \right] u_{nl}(r) = 0. \quad (3.7)$$

introducing a new parametr

$$\tilde{l} = l + \frac{D-3}{2},$$

Eq.(3.7) takes the form

$$\frac{d^2 u_{nl}(r)}{dr^2} + \frac{2\mu}{\hbar^2} \left[E + \frac{V_0}{1 + \exp\left(\frac{r-R_0}{a}\right)} - \frac{\hbar^2 \tilde{l}(\tilde{l} + 1)}{2\mu r^2} \right] u_{nl}(r) = 0. \quad (3.8)$$

Equation (3.8) has the same form as the equation for a particle in one dimension, except for two important differences. First, there is a repulsive effective potential proportional to the eigenvalue of $\hbar^2 \tilde{l}(\tilde{l} + 1)$. Second, the radial function must satisfy the boundary conditions $u(0) = 0$ and $u(\infty) = 0$.

It is sometimes convenient to define in Eq.(3.8) the effective potential in the form:

$$V_{eff}(r) = V(r) + \frac{\hbar^2 \tilde{l}(\tilde{l} + 1)}{2\mu r^2}. \quad (3.9)$$

Then, the radial Schrödinger equation given by Eq.(3.8) takes the form

$$\frac{d^2 u_{nl}(r)}{dr^2} + \frac{2\mu}{\hbar^2} [E - V_{eff}(r)] u_{nl}(r) = 0. \quad (3.10)$$

If in Eq.(3.1) introduce the notations

$$x = \frac{r - R_0}{R_0}, \quad \alpha = \frac{R_0}{a},$$

then the Woods-Saxon potential is given by the expression

$$V_{WS} = -\frac{V_0}{1 + \exp(\alpha x)}.$$

The effective potential together with the WS potential for $l \neq 0$ can be written as

$$V_{eff}(r) = V_l(r) + V_{WS}(r) = \frac{\hbar^2 \tilde{l}(\tilde{l} + 1)}{2\mu r^2} - \frac{V_0}{1 + \exp(\alpha x)}. \quad (3.11)$$

It is known that the Schrödinger equation cannot be solved exactly for this potential at the value $\tilde{l} \neq 0$ using the standard methods as SUSY and NU. From Eq.(3.11) it is seen

that the effective potential is a combination of the exponential and inverse square potentials, which cannot be solved analytically. Therefore, in order to solve this problem we can take the most widely used and convenient for our purposes Pekeris approximation [3-8,]. This approximation is based on the expansion of the centrifugal barrier in a series of exponentials depending on the internuclear distance, taking into account terms up to second order, so that the effective l -dependent potential preserves the original form. It should be pointed out, however, that this approximation is valid only for low vibrational energy states. By changing the coordinates $x = \frac{r-R_0}{R_0}$ or $r = R_0(1+x)$, the centrifugal potential is expanded in the Taylor series around the point $x = 0$ ($r = R_0$)

$$V_l(r) = \frac{\hbar^2 \tilde{l}(\tilde{l}+1)}{2\mu r^2} = \frac{\hbar^2 \tilde{l}(\tilde{l}+1)}{2\mu R_0^2} \frac{1}{(1+x)^2} = \delta (1 - 2x + 3x^2 - 4x^3 + \dots), \quad (3.12)$$

where $\tilde{\delta} = \frac{\hbar^2 \tilde{l}(\tilde{l}+1)}{2\mu R_0^2}$.

According to the Pekeris approximation, we shall replace potential $V_l(r)$ with expression

$$V_l^*(r) = \tilde{\delta} \left(C_0 + \frac{C_1}{1 + \exp \alpha x} + \frac{C_2}{(1 + \exp \alpha x)^2} \right). \quad (3.13)$$

In order to define the parameters C_0 , C_1 and C_2 , we also expand this potential in the Taylor series around the point $x = 0$ ($r = R_0$):

$$V_l^*(x) = \tilde{\delta} \left[\left(C_0 + \frac{C_1}{2} + \frac{C_2}{4} \right) - \frac{\alpha}{4} (C_1 + C_2) x + \frac{\alpha^2}{16} C_2 x^2 + \frac{\alpha^3}{48} (C_1 + C_2) x^3 - \frac{\alpha^4}{96} C_2 x^4 + \dots \right]. \quad (3.14)$$

Comparing equal powers of x Eqs.(3.12) and (3.14), we obtain the constants C_0 , C_1 and C_2 :

$$C_0 = 1 - \frac{4}{\alpha} + \frac{12}{\alpha^2}, \quad C_1 = \frac{8}{\alpha} - \frac{48}{\alpha^2}, \quad C_2 = \frac{48}{\alpha^2}.$$

Now, the effective potential after Pekeris approximation becomes equal to

$$V_{eff}^*(x) = V_l^*(x) + V_{WS}(x) = \tilde{\delta} C_0 - \frac{V_0 - \tilde{\delta} C_1}{1 + \exp(\alpha x)} + \frac{\tilde{\delta} C_2}{(1 + \exp(\alpha x))^2}. \quad (3.15)$$

Instead of solving the radial Schrödinger equation for the effective Woods-Saxon potential $V_{eff}(r)$ given by Eq.(3.11), we now solve the radial Schrödinger equation for the new effective potential $V_{eff}^*(r)$ given by Eq.(3.15) obtained using the Pekeris approximation. Having inserted this new effective potential into Eq.(3.10), we obtain

$$\frac{d^2 u_{nl}(r)}{dr^2} + \frac{2\mu}{\hbar^2} \left[E - \tilde{\delta} C_0 + \frac{V_0 - \tilde{\delta} C_1}{1 + \exp\left(\frac{r-R_0}{a}\right)} - \frac{\tilde{\delta} C_2}{\left(1 + \exp\left(\frac{r-R_0}{a}\right)\right)^2} \right] u_{nl}(r) = 0. \quad (3.16)$$

We use the following dimensionless notations:

$$\epsilon^2 = -\frac{2\mu(E - \tilde{\delta}C_0)a^2}{\hbar^2}; \quad \beta^2 = \frac{2\mu(V_0 - \tilde{\delta}C_1)a^2}{\hbar^2}; \quad \gamma^2 = \frac{2\mu\tilde{\delta}C_2a^2}{\hbar^2}, \quad (3.17)$$

with real $\epsilon > 0$ for bound states; β and γ are real and positive.

If we rewrite Eq.(3.16) by using a new variable of the form

$$z = \left(1 + \exp\left(\frac{r - R_0}{a}\right)\right)^{-1},$$

we obtain

$$u''(z) + \frac{1 - 2z}{z(1 - z)}u'(z) + \frac{-\epsilon^2 + \beta^2z - \gamma^2z^2}{(z(1 - z))^2}u(z) = 0, \quad (0 \leq z \leq 1), \quad (3.18)$$

with $\tilde{\tau}(z) = 1 - 2z$; $\sigma(z) = z(1 - z)$; $\tilde{\sigma}(z) = -\epsilon^2 + \beta^2z - \gamma^2z^2$.

In the NU-method, the new function $\pi(z)$ is

$$\pi(z) = \pm \sqrt{\epsilon^2 + (k - \beta^2)z - (k - \gamma^2)z^2}. \quad (3.19)$$

The constant parameter k can be found employing the condition that the expression under the square root has a double zero, i.e., its discriminant is equal to zero. Hence, there are two possible functions for each k :

$$\pi(z) = \pm \begin{cases} \left(\epsilon - \sqrt{\epsilon^2 - \beta^2 + \gamma^2}\right)z - \epsilon, & \text{for } k = \beta^2 - 2\epsilon^2 + 2\epsilon\sqrt{\epsilon^2 - \beta^2 + \gamma^2}, \\ \left(\epsilon + \sqrt{\epsilon^2 - \beta^2 + \gamma^2}\right)z - \epsilon, & \text{for } k = \beta^2 - 2\epsilon^2 - 2\epsilon\sqrt{\epsilon^2 - \beta^2 + \gamma^2}. \end{cases} \quad (3.20)$$

According to the NU-method, from the four possible forms of the polynomial $\pi(z)$ we select the one for which the function $\tau(z)$ has the negative derivative and root lies in the interval (0,1). Therefore, the appropriate functions $\pi(z)$ and $\tau(z)$ have the following forms:

$$\pi(z) = \epsilon - \left(\epsilon + \sqrt{\epsilon^2 - \beta^2 + \gamma^2}\right)z, \quad (3.21)$$

$$\tau(z) = 1 + 2\epsilon - 2\left(1 + \epsilon + \sqrt{\epsilon^2 - \beta^2 + \gamma^2}\right)z, \quad (3.22)$$

and

$$k = \beta^2 - 2\epsilon^2 - 2\epsilon\sqrt{\epsilon^2 - \beta^2 + \gamma^2}. \quad (3.23)$$

Then, the constant $\lambda = k + \pi'(z)$ is written as

$$\lambda = \beta^2 - 2\epsilon^2 - 2\epsilon\sqrt{\epsilon^2 - \beta^2 + \gamma^2} - \epsilon + \sqrt{\epsilon^2 - \beta^2 + \gamma^2}. \quad (3.24)$$

An alternative definition of λ_n (Eq.(2.9)) is

$$\lambda = \lambda_n = 2 \left(\epsilon + \sqrt{\epsilon^2 - \beta^2 + \gamma^2} \right) n + n(n+1). \quad (3.25)$$

Having compared Eqs.(3.19) and (3.20)

$$\beta^2 - 2\epsilon^2 - 2\epsilon\sqrt{\epsilon^2 - \beta^2 + \gamma^2} - \epsilon - \sqrt{\epsilon^2 - \beta^2 + \gamma^2} = 2 \left(\epsilon + \sqrt{\epsilon^2 - \beta^2 + \gamma^2} \right) n + n(n+1), \quad (3.26)$$

we obtain

$$\epsilon + \sqrt{\epsilon^2 - \beta^2 + \gamma^2} + n + \frac{1}{2} - \frac{1}{2}\sqrt{1 + 4\gamma^2} = 0, \quad (3.27)$$

or

$$\epsilon + \sqrt{\epsilon^2 - \beta^2 + \gamma^2} = n'. \quad (3.28)$$

After some elementary calculus, one sees that

$$\epsilon - \sqrt{\epsilon^2 - \beta^2 + \gamma^2} = \frac{\beta^2 - \gamma^2}{n'}. \quad (3.29)$$

Here

$$n' = -n + \frac{\sqrt{1 + 4\gamma^2} - 1}{2}, \quad (3.30)$$

n being the radial quantum number ($n = 0, 1, 2, \dots$). From Eq.(3.28) and (3.29), we find

$$\epsilon = \frac{1}{2} \left(n' + \frac{\beta^2 - \gamma^2}{n'} \right) \quad (3.31)$$

and

$$\sqrt{\epsilon^2 - \beta^2 + \gamma^2} = \frac{1}{2} \left(n' - \frac{\beta^2 - \gamma^2}{n'} \right) \quad (3.32)$$

Because for the bound states $\epsilon > 0$ and Eqs.(3.28, 3.29, 3.31, 3.32) we get

$$n' > 0 \quad (3.33)$$

and

$$0 < \beta^2 - \gamma^2 < n'^2. \quad (3.34)$$

If $n' > 0$, there exist bound states, otherwise, there are no bound states at all. By using Eq.(3.30) this relation can be recast into the form

$$0 \leq n < \frac{\sqrt{1 + 4\gamma^2} - 1}{2}, \quad (3.35)$$

i.e. it gives the finite coupling value.

If $\beta^2 - \gamma^2 > 0$, there exists bound states; otherwise there are no bound states. Inequality, which obtained after substituting β, γ, C_1, C_2 into Eq.(3.34), gives the definite coupling value for the potential depth V_0 :

$$V_0 > \frac{\hbar^2 \tilde{l}(\tilde{l}+1)a}{2\mu R_0^3} \quad (3.36)$$

After substituting $\gamma, \tilde{\delta}, C_2$ into Eq.(3.36), we find

$$0 \leq n < \frac{\sqrt{1 + \frac{192a^4 \tilde{l}(\tilde{l}+1)}{R_0^4}} - 1}{2} \quad (3.37)$$

The exact energy eigenvalues of the Schrödinger equation with the Woods-Saxon potential are derived as

$$E_{nl}^{(D)} = \tilde{\delta} C_0 - (V_0 - \tilde{\delta} C_1) \left(\frac{n'^2 + \beta^2 - \gamma^2}{2\beta n'} \right)^2. \quad (3.38)$$

Substituting the values of $\tilde{\delta}, C_0, C_1, C_2, n', \beta$ and γ into (3.38), one can find E_{nl}

$$E_{nl}^{(D)} = \frac{\hbar^2 \tilde{l}(\tilde{l}+1)}{2\mu R_0^2} \left(1 + \frac{12a^2}{R_0^2} \right) - \frac{\hbar^2}{2\mu a^2} \left\{ \frac{\left[\sqrt{1 + \frac{192\tilde{l}(\tilde{l}+1)a^4}{R_0^4}} - 2n - 1 \right]^2}{16} + \frac{4 \left[\frac{\mu a^2 V_0}{\hbar^2} - \frac{4\tilde{l}(\tilde{l}+1)a^3}{R_0^3} \right]^2}{\left[\sqrt{1 + \frac{192\tilde{l}(\tilde{l}+1)a^4}{R_0^4}} - 2n - 1 \right]^2} + \frac{\mu V_0 a^2}{\hbar^2} \right\} \quad (3.39)$$

In three-dimensions, the case when we set the parameter value $D = 3$ in Eq.(3.39), then we obtain the energy of the Woods-Saxon potential

$$E_{nl} = \frac{\hbar^2 l(l+1)}{2\mu R_0^2} \left(1 + \frac{12a^2}{R_0^2} \right) - \frac{\hbar^2}{2\mu a^2} \left\{ \frac{\left[\sqrt{1 + \frac{192l(l+1)a^4}{R_0^4}} - 2n - 1 \right]^2}{16} + \frac{4 \left[\frac{\mu a^2 V_0}{\hbar^2} - \frac{4l(l+1)a^3}{R_0^3} \right]^2}{\left[\sqrt{1 + \frac{192l(l+1)a^4}{R_0^4}} - 2n - 1 \right]^2} + \frac{\mu V_0 a^2}{\hbar^2} \right\} \quad (3.40)$$

which is identical to the one obtained in Ref.[8].

If both conditions (3.36) and (3.37) are satisfied simultaneously, the bound states exist. For very large V_0 the l -dependent effective potential has the same form as the potential with $l = 0$. Whenever $D = 3$, the bound states of the system do not exist in $l = 0$ state, because (3.36) and (3.37) inequalities are not satisfied. Hence, the radial Schrödinger equation for the standard Woods-Saxon potential with zero angular momentum has no bound state. Furthermore, whenever $D > 3$, the bound states of the system exist in $l = 0$ state. Thus,

the energy spectrum equation (3.39) is limited, i.e. we have only the finite number of energy eigenvalues.

According to Eq.(3.39), the energy eigenvalues depend on the depth of the potential V_0 , the width of the potential R_0 , surface thickness a , and the parameter D . If constraints imposed on n and V_0 satisfied, the bound states appear. From Eq.(3.36), it is seen that the potential depth decreases when the parameter R_0 increases and the parameter a decreases, and vice versa. Therefore, one can say that the bound states exist within this potential.

In addition, we have seen that there are some restrictions on the potential parameters for the bound state solutions within the framework of quantum mechanics. That is, when the values of the parameters V_0 and n satisfy the conditions (3.36) and (3.37), we obtain the bound states. We also point out that the exact results obtained for the standard Woods-Saxon potential may have some interesting applications for studying different quantum mechanical and nuclear scattering problems. Consequently, the found wave functions are physical ones.

Now, we are going to determine the radial eigenfunctions of this potential. Having substituted $\pi(z)$ and $\sigma(z)$ into Eq.(2.4) and then solving first-order differential equation, one can find the finite function $\Phi(z)$ in the interval $[0, 1]$

$$\Phi(z) = z^\epsilon (1 - z)^{\sqrt{\epsilon^2 - \beta^2 + \gamma^2}}. \quad (3.41)$$

It is easy to find the second part of the wave function from the definition of weight function:

$$\rho(z) = z^{2\epsilon} (1 - z)^{2\sqrt{\epsilon^2 - \beta^2 + \gamma^2}}, \quad (3.42)$$

and substituting into Rodrigues relation (2.4), we get

$$y_n(z) = B_n z^{-2\epsilon} (1 - z)^{-2\sqrt{\epsilon^2 - \beta^2 + \gamma^2}} \frac{d^n}{dz^n} \left[z^{n+2\epsilon} (1 - z)^{n+2\sqrt{\epsilon^2 - \beta^2 + \gamma^2}} \right], \quad (3.43)$$

where B_n is the normalization constant and its value is $\frac{1}{n!}$ [17]. Then, y_n is given by the Jacobi polynomials:

$$y_n(z) = P_n^{(2\epsilon, 2\sqrt{\epsilon^2 - \beta^2 + \gamma^2})}(1 - 2z),$$

where

$$P_n^{(\alpha, \beta)}(1 - 2z) = \frac{1}{n!} z^{-\alpha} (1 - z)^{-\beta} \frac{d^n}{dz^n} \left[z^{n+\alpha} (1 - z)^{n+\beta} \right].$$

The corresponding $u_{nl}(z)$ radial wave functions are found to be

$$u_{nl}(z) = C_{nl} z^\epsilon (1-z)^{\sqrt{\epsilon^2 - \beta^2 + \gamma^2}} P_n^{(2\epsilon, 2\sqrt{\epsilon^2 - \beta^2 + \gamma^2})}(1-2z), \quad (3.44)$$

where C_{nl} is a new normalization constant determined using $\int_0^\infty [u_{nl}(r)]^2 dr = 1$.

IV. CONCLUSION

In this paper, we have analytically calculated energy eigenvalues of the bound states and corresponding eigenfunctions in the new exactly solvable Woods-Saxon potential. The energy eigenvalue expression for Woods-Saxon potentials is given by Eq.(3.39). As it should be expected (see Eq.(3.39)), for any given set of parameters V_0, R_0, a and D , the energy levels of standard Woods-Saxon potential are positive. We can conclude that our results are interesting not only for pure theoretical physicist but also for experimental physicist, because the results are exact and more general and useful to study nuclear scattering.

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